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Gleason-Type Theorem for Linear Spaces over the Field of Four Elements

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We prove an analog of the famous Gleason theorem for additive functions on the orthomodular poset of all projections defined on an n-dimensional linear space over the field consisting of four elements. An essential part of the proof consists in a computer calculation.

1. INTRODUCTION

Let F and $G \subset F$ be two fields, X be a finite-dimensional linear space over F, and $\mathfrak{P}(X)$ be the orthomodular set of all (linear) projections on X. Let us consider a G-valued measure μ: $\mathfrak{P}(X) \to G$ which is additive, i.e., $\mu(P + Q) = \mu(P) + \mu(Q)$ for $P, Q \in \mathfrak{P}(X)$ with $PQ = QP = 0$. Our goal is to extend (when possible) the function μ to a G-valued additive functional defined on the linear space over G generated by $\mathfrak{P}(X)$.

Such a theorem was proved when G and F coincide with the set $\mathbf{Q}$ of rationals or a residue field (Mushtari, 1995). But, for extending the result of (Mushtari, 1989) to some classes of topological linear spaces we need to consider the case $\mathbf{G} = \mathbf{Q}$ and F being an extension of $\mathbf{Q}$. Unfortunately, we have not managed to prove such a theorem. In this paper, we try to elaborate the technique we need in a simpler case.

2. THE THEOREM

Theorem 1. Let X be a finite-dimensional linear space over the field F_4 consisting of 4 elements, $\dim X \geq 3$, F_2 be the field $\{0, 1\}$, and μ: $\mathfrak{P}(X) \to$

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F_2 be a measure. Then μ admits an F_2 linear extension to the set $UL(X)$ of all F_4-linear operators with the trace belonging to F_2.

Proof. The proof essentially depends upon the results of a computer calculation. We omit the calculation and only present its result. As in the proof of the classical Gleason theorem, we only have to consider the 3-dimensional case. We notice also (Mushtari, 1995) that we have to really prove that the restriction of μ to the set of all 1-dimensional projections is uniquely determined by the values of μ on a fixed Hamel basis in the F_2-linear space $UL(X)$. Note that the F_2-dimension of $UL(X)$ is equal to 17.

Step 1. The case $X = F_2^3$. The number of one-dimensional projections equals 28. The number of all bases equals 28. Every such basis $\{e, f, g\}$ (together with its biorthogonal one $\{e', f', g'\}$ defines the three mutually orthogonal projections $e'. \otimes e{:}x \mapsto e'(x)e, f' \otimes f'$, and $g' \otimes g$. Since $\mu(e' \otimes e) + \mu(f' \otimes f) + \mu(g' \otimes g) = \mu(\mathrm{Id})$, every such basis defines an equation for μ. It can be readily verified that the rank of this system is equal to 19. So, the values of μ on some nine linearly independent projections determine the μ on $\mathfrak{P}\ (F_2^3)$ uniquely.

Step 2. The case $X = F_4^3$. What does Step 1 provide? Let us consider an arbitrary basis $\{e, f, g\}$ in F_4^3 with the biorthogonal $\{e', f', g'\}$ and the set $\mathfrak{P}_2(\{e, f, g\})$ of all projections P of the form $P = (\alpha'_1 e' + \alpha'_2 f' + \alpha'_3 g') \otimes (\alpha_1 e + \alpha_2 f + \alpha_3 g)$, where all α_i belong to F_2. By Step 1, μ is uniquely determined on $\mathfrak{P}_2\ (\{e, f, g\})$ by its values on nine basic projections, for example:

$$P_1 = e' \otimes e, \qquad P_2 = (e' + f') \otimes e, \qquad P_3 = (e' + g') \otimes e$$

$$P_4 = (f' + e') \otimes f, \qquad P_5 = f' \otimes f, \qquad P_6 = (f' + g') \otimes f$$

$$P_7 = (g' + e') \otimes g, \qquad P_8 = (g' + f') \otimes g, \qquad P_9 = g' \otimes g$$

Now we have to prove that the values of μ on $\{P_i\text{: } i \leq 9\}$, where $\{e, f, g\}\ \varepsilon\ F_2^3$ and on some additional eight projections uniquely determine μ on the whole $\mathfrak{P}_2(F_4^3)$.

Denote $F_4 = \{0, 1, i, i^{-1}\}$, wherein $i^2 = i + 1 = i^{-1}, i^{-2} = i^{-1} + 1 = i$. Let us consider all bases in F_2^3. To any such basis $\{e, f, g\}$, we add the six bases in F_4^3, namely

$$\{ie, f, g\}, \{e, if, g\}, \{e, f, ig\}, \{e, if, ig\}, \{ie, f, ig\}, \{ie, if, g\}$$

and add the set $\mathfrak{P}_2\ (\{e, f, g\})$ of 12 projections, namely,

$$\tilde{P}_1 = (i^{-1}e' + f') \otimes ie, \qquad \tilde{P}_2 = (i^{-1}e' + g') \otimes ie, \qquad \tilde{P}_3 = (i^{-1}f' + e') \otimes if$$

$$\tilde{P}_4 = (i^{-1}f' + g') \otimes if, \qquad \tilde{P}_5 = (i^{-1}g' + e') \otimes ig, \qquad \tilde{P}_6 = (i^{-1}g' + f') \otimes ig$$

$$\tilde{P}_7 = i^{-1}e' \otimes (ie + f), \qquad \tilde{P}_8 = i^{-1}e' \otimes (ie + g), \qquad \tilde{P}_9 = i^{-1}f' \otimes (if + e)$$

$$\tilde{P}_{10} = i^{-1}f' \otimes (if + g), \qquad \tilde{P}_{11} = i^{-1}g' \otimes (ig + e), \qquad \tilde{P}_{12} = i^{-1}g' \otimes (ig + f)$$

We have the six sets $\mathfrak{P}_2(\{ie, f, g\})$, $\mathfrak{P}_2(\{e, if, g\})$, . . . , $\mathfrak{P}_2(\{ie, if, g\})$. It is easy to see that bases (of nine elements) in these sets can be chosen so that all their elements belong to $\{P_i: i \leq 9\} \cup \mathfrak{P}_2$ $(\{e, f, g\})$. We denote by $\mathfrak{P}_2(\{e, f, g\})$ the union of all these six sets and denote $\mathfrak{P}_2(\{e, f, g\}) = \mathfrak{P}_2(\{e, f, g\}) \cup \mathfrak{P}_2(\{e, f, g\})$. We denote by $\mathfrak{P}$ the direct sum of all sets $\mathfrak{P}_2(\{e, f, g\})$ for all bases in F_2^3. There are 28 bases in F_2^3. The 12 basic elements correspond to every basis. So, $\mathfrak{P}$ contains 28 × 12 elements.

Now, we describe the equations which connect the values of μ on $\mathfrak{P}$. Let us consider a projection $P = (i^{-1}\, a' + b') \otimes ia$, wherein $a, b \in F_2^3$, $a' \in (F_2^3)'$, $b'(a) = 0$. If P belongs to $\mathfrak{P}_2(\{x, y, z\})$, then $\mu(P)$ is uniquely determined by the values of μ on $\mathfrak{P}_2(\{x, y, z\})$ (and on the set $\{P_1: i \leq 9\}$). It is easy to see that P belongs to seven sets of projections $\mathfrak{P}_2(x_1, x_2, x_3)$, where either $a = x_i$, $a' = x_i'$, $b \in \mathrm{Lin}(x_j, x_k: j, k \neq i)$, or $b = x_i$, $a \in \mathrm{Lin}(x_j, x_k: j, k \neq i)$, $a' \,\varepsilon\, \mathrm{Lin}(x_j', x_k': j, k \neq i)$. The P may be represented as $P = [(a + b) + ib] \otimes a'$ and thus belongs to seven other sets, $\mathfrak{P}_2\{(x, y, z)\}$. We can calculate $\mu(P)$ by using 14 different bases. This gives 13 equations which connect the values of μ on $\mathfrak{P}$. Eventually, we obtain 13 × 84 different equations. The computer calculation shows that the rank of the system is equal to 328.

We now summarize the result of computing. Denote

$$\mathfrak{B}_2 = \cup\, \{\mathfrak{B}_2(\{e, f, g\}): \{e, f, g\} \text{ is a basis in } F_2^3\}, \qquad \mathfrak{B}_2 = \mathfrak{B}_2 \cup \mathfrak{B}_2$$

The μ on $\mathfrak{P}_2$ is uniquely determined by the values of μ on every basis of 17 elements [9 basic elements in $\mathfrak{P}_2$ and 8 (= 336 − 328) basic elements in $\mathfrak{P}$].

Since $\{1, i\}$ is a basis in F_4, every one-dimensional projection on F_4^3 admits the representative $P = (i^{-1}\, a' + b') \otimes (ia + b)$, where $a, b \in F_2^3$ and $a', b' \in (F_2^3)'$ satisfy one of the following:

(i) $b'(a) = a'(b) = b'(b) = 0, a'(a) = 1$
(ii) $b'(a) = a'(b) = b'(b) = 0, b'(b) = 1$
(iii) $b'(a) = a'(b) = b'(b) = a'(a) = 1$
(iv) $b'(a) = a'(b) = 1, b'(b) = a'(a) = 0$

It is easy to see that $\mathfrak{P}_2$ consists of all projections $(i^{-1}\, a' + b') \otimes (ia + b)$ satisfying (i) or (ii). The case (iii) can be reduced to (i) or (ii). Actually, by using $i = i^{-1} + 1$ and $i^{-1} = i + 1$, we obtain

$$P = (i^{-1}a' + b') \otimes (ia + b) = [i^{-1}(a' + b') + a'] \otimes [i(a + b) + a]$$

The case (iv) is a little more difficult. It is more convenient to represent P as $P = (i^{-1} y' + x') \otimes (ix + y)$, where $x, y \in F_2^3$, $x', y' \in (F_2^3)'$, $y'(x) = x'(y) = 0$, and $x'(x) = y'(y) = 1$. Obviously, in this case, P is orthogonal to $P_1 = (i^{-1} x' + y') \otimes (iy + x)$. Consider the basis $\{x, y, z\}$ in F_2^3 with the biorthogonal basis $\{x', y', z'\}$. Also consider the basis $\{\xi = ix + y, \eta = iy + x, \zeta = z\}$ in F_4^3 with the biorthogonal basis $\{\xi' = i^{-1} y' + x', \eta' = i^{-1}x' + y', \zeta' = z'\}$. Now, we construct a basis in $\mathfrak{P}_2(\{e, f, g\})$ whose elements belong to $\mathfrak{P}_2$. This proves that $\mu(P)$ is uniquely determined by the values of μ on $\mathfrak{P}_2$. Actually, the basis we need consists of the following projections:

$$P_1 = (\xi' + \eta') \otimes \xi = (x' + y') \otimes [i(x + y) + x]$$

$$P_2 = (\xi' + \eta') \otimes \eta = (x' + y') \otimes [i(x + y) + y]$$

$$P_3 = (\xi' + \zeta') \otimes (\eta + \zeta) = [i^{-1}y' + x' + z'] \otimes [iy + x + z]$$

$$P_4 = (\eta' + \zeta') \otimes (\xi + \zeta) = [i^{-1}x' + y' + z'] \otimes [ix + y + z]$$

$$P_5 = \zeta' \otimes \zeta$$

$$P_6 = (\xi' + \zeta') \otimes \zeta$$

$$P_7 = (\zeta' \otimes (\xi + \zeta)$$

$$P_8 = (\eta' + \zeta') \otimes \zeta$$

$$P_9 = \zeta' \otimes (\eta + \zeta)$$

(Observe that we have made use of the third component).

This proves the theorem in the case $n = 3$. The case of an arbitrary n can be reduced to $n = 3$ as in Mushtari (1995).

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